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A FOURTH-ORDER NONLINEAR ITERATIVE METHOD IN BANACH SPACES

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Abstract—In this short paper, we establish an Ostrowski-Kantorovich convergence theorem [1–5] and give an expression for the error bound of a fourth order method in Banach spaces.

1. INTRODUCTION

We introduce a new iterative method for solving nonlinear operator equations in Banach spaces. Comparing with the classical Newton iteration, this new method can save one evaluation of $P(X)$ and an inverse computation in each step. Also we provide sufficient conditions and give a representation for the error bound of the method.

2. A FOURTH ORDER ITERATIVE METHOD

Let $P(X) : D_o \subset X_B \longrightarrow Y_B$, where X_B and Y_B are real or complex Banach spaces and D_o is an open domain in X_B. Assume that P has first order continuous Frechet derivative on D_o and $P'(X)^{-1}$ exists. We define the method as follows for all $n \geqq 0$:

$$\begin{aligned} Y_n &= X_n - P'(X_n)^{-1}P(X_n) \\ H(X_{n'}Y_n) &= P'(X_n)^{-1}\left\{P'\left[X_n + \frac{2}{3}(Y_n - X_n)\right] - P'(X_n)\right\} \\ X_{n+1} &= Y_n - \frac{3}{4}H(X_{n'}Y_n)\left[I - \frac{3}{2}H(X_{n'}Y_n)\right](Y_n - X_n). \end{aligned} \tag{2.1}$$

Under the standard Ostrowski-Kantorovich assumptions, we can show that iteration (2.1) is well-defined. We also provide sufficient convergence conditions and derive a representation for the error bound. The main result is the following theorem.

2. THE CONVERGENCE THEOREM AND ERROR BOUND

THEOREM 3.1. *Let $P(X) : D_o \subset X_B \longrightarrow Y_B$, where X_B, Y_B are real or complex Banach spaces and D_o is an open convex domain.*

Assume that P has 2nd order continuous Frechet derivatives on D_o and satisfies the following conditions:

$$\|P''(X)\| \leqq M, \quad \|P''(X) - P''(Y)\| \leqq N\|X - Y\|, \tag{3.2}$$

for all X, Y in D_o. For given an initial value $X_0 \in D_o$, assume that $P'(X_0)^{-1}$ exists and satisfies

$$\|P'(X_0)^{-1}\| \leqq \beta, \quad \|Y_0 - X_0\| \leqq \eta. \tag{3.3}$$

Set

$$\left\{1M^2 + \frac{N}{6\beta}\right\}^{1/2} \leqq K, \tag{3.4}$$

$$h = K\beta\eta \leqq 0.46568, \tag{3.5}$$

$$\overline{S(X_0, r_1)} \subset D_o, \tag{3.6}$$

where $\overline{S(x,r)} = \{x' \epsilon X \mid |x' - x| \leqq r\}$ and

$$g(t) = \frac{1}{2}KT^2 - \frac{1}{\beta}t + \frac{\eta}{\beta}, \tag{3.7}$$

$$r_1 = \frac{1 - \sqrt{1 - 2h}}{h}\eta, \tag{3.8}$$

$$\theta = \frac{1 - \sqrt{1 - 2h}}{1 + \sqrt{1 - 2h}}. \tag{3.9}$$

Here r_1 is the smallest root of equation (3.7). Then the iterative procedure (2.1) is well-defined and convergent. Also $X_n, Y_n \epsilon\ \overline{S(X_0, r_1)}$ for all $n \geqq 0$. The limit X^* is a solution of the equation $P(X) = 0$. In addition, we have the following error estimates.

$$\|X_n - X^*\| \leq r_1 - t_n \leq \frac{(1 - \theta^2)\eta}{1 - \frac{1}{\sqrt[3]{5}}\left[\sqrt[3]{5}\ \theta\right]^{4^n}} \left[\sqrt[3]{5}\ \theta\right]^{4^n - 1} \tag{3.10}$$

for all $n \geqq 0$. To produce the sequence $\{t_n\}n \geq 0$, replace Y_n, X_n and P in (2.1) by S_n, t_n and g, respectively, for all $n \geq 0$.

REMARKS.

(a) The constant $(h \leqq) 0.46568\ldots$ results from the solution of the inequality

$$\sqrt[3]{5}\ \theta = \sqrt[3]{5}\,\frac{1 - \sqrt{1 - 2h}}{1 + \sqrt{1 - 2h}} \leq 1.$$

(b) Our results can be used to solve multilinear operator equations of the form

$$P_k(X) = M_k X^k + M_{k-1} X^{k-1} + \cdots + M_1 X + M_0$$

and each M_i is a linear symmetric operator. In particular, for $P_2(X) = M_2X^2 + M_1X + M_0$ we have $P_2'(X) = 2M_2X + M_1$ and $P_2''(X) = 2M_2$. That is in conditions (3.2), we can set $M = 2\|M_2\|$ and $N = 0$. Equations of the form $P_2(X) = 0$ have very important applications in radioactive transfer [6].

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